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On reduced equations in the Hamiltonian theory of weakly nonlinear surface waves

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Many studies of weakly nonlinear surface waves are based on so-called reduced integrodifferential equations. One of these is the widely used Zakharov four-wave equation for purely gravity waves. But the reduced equations now in use are not Hamiltonian despite the Hamiltonian structure of exact water wave equations. This is entirely due to shortcomings of their derivation. The classical method of canonical transformations, generalized to the continuous case, leads automatically to reduced equations with Hamiltonian structure. In this paper, attention is primarily paid to the Hamiltonian reduced equation describing the combined effects of four- and five-wave weakly nonlinear interactions of purely gravity waves. In this equation, for brevity called five-wave, the non-resonant quadratic, cubic and fourth-order nonlinear terms are eliminated by suitable canonical transformation. The kernels of this equation and the coefficients of the transformation are expressed in explicit form in terms of expansion coefficients of the gravity-wave Hamiltonian in integral-power series in normal variables. For capillary–gravity waves on a fluid of finite depth, expansion of the Hamiltonian in integral-power series in a normal variable with accuracy up to the fifth-order terms is also given.

1. Introduction

The Hamiltonian description of surface waves, first suggested by Zakharov (1968), more than twenty years ago (see also Broer 1974; Miles 1977), has placed the problem of surface waves in a line of numerous problems of nonlinear dispersive waves in continuous conservative media (Zakharov 1974). In this method the evolution equations for irrotational surface waves are presented in the form of the canonical Hamilton equations in which the Hamiltonian is the total energy of the waves and the canonically conjugate variables are the free-surface elevation and the velocity potential evaluated at the surface.

Advantages of the Hamiltonian approach are now well-known. In particular, specific features of a medium turn out to be, in large part, unessential; all versions of the perturbation theory are considerably simplified and standardized; results of calculations obtained for a particular medium are easily assigned a general-physics meaning. One of the consequences of the Hamiltonian approach is an integro-differential evolution equation of standard form for a so-called normal variable (or, in other terminology, complex wave amplitude) a related by a transformation of Fourier type with 'natural' physical variables. The general structure of this equation is the same for waves of a different physical nature in nonlinear dispersive media, allowing the introduction of canonical variables, and specific features of waves are absorbed by coefficients of this equation and, in particular, by the dispersion relation of linear waves. From this evolution equation for a one usually derives somewhat more simple integrodifferential equations for an auxiliary variable b, which we term 'reduced equations'. For the case of purely gravity waves an example is an equation cubically nonlinear in b describing weakly nonlinear four-wave interactions, often referred to as the Zakharov equation. The reduced equations usually serve as a starting point for the study of wave instabilities, long-time wave evolution, derivation of transfer (kinetic) equations for spectrum of random wave field and for other applications.

But the reduced equations now employed have a fundamental shortcoming: they are not Hamiltonian (and thus non-conservative), although the exact water wave equations form a Hamiltonian system. The fact that the Zakharov equation in its original version (Zakharov 1966, 1968; see also Crawford, Saffman & Yuen 1980, where some omissions in the earlier publications were corrected) is not Hamiltonian was pointed out by Caponi, Saffman & Yuen (1982). This has puzzled investigators up to now, though this non-conservative reduced equation remains in use. There was even the supposition that this non-conservativity is completely related to the retention of only cubic terms in the Zakharov equation, and that retaining the fourth- and the higher-order terms must lead to energy conservation with increasing accuracy (Stiassnie & Shemer 1987).

In fact this non-conservativity is caused solely by shortcomings of the techniques used for derivation of the Zakharov reduced equation. There are at least two causes for the non-Hamiltonian structure of this equation. Firstly, in some of the papers the evolution equation for a was derived from the original hydrodynamical equations of surface wave theory, but not from the Hamiltonian formulation, and therefore the coefficients of this equation do not satisfy proper symmetry conditions (see e.g. Yuen & Lake 1982; Stiassnie & Shemer 1984) expressing the Hamiltonian structure of the system. This leads to apparent violation of the Hamiltonian structure. Another, more serious, cause for violation of the Hamiltonian structure of the reduced equation for b is connected with the techniques which have been used for its derivation from the evolution equation for a. This reduced equation was first derived by Zakharov (1966, 1968) by heuristic considerations, which later were somewhat formalized by Crawford *et al.* (1980). Both methods lead to the non-Hamiltonian reduced equation not conserving energy.

However, there is another technique for derivation of the reduced equations, quite natural within the framework of the Hamiltonian formulation. It is the classical method of canonical transformations from discrete mechanics generalized to the continuous case. In this method, the variables a and b are related by a canonical transformation prescribed in the form of an integral-power series in b. Coefficients of this transformation can be chosen so as to eliminate 'unimportant' non-resonant terms from the Hamiltonian expressed in terms of the new variable b. The resulting 'reduced Hamiltonian' yields, through the canonical Hamilton equation for b, the Hamiltonian reduced equation. This general idea was mentioned by Zakharov (1974) and West (1981) but without detailed elaboration.

In this paper, most attention is given to the reduced equations themselves and not their possible applications. In §2, we summarize the basic equations of the Hamiltonian theory of surface waves and describe the general idea of using canonical transformations for derivation of the reduced equations. In §3, we discuss the conditions under which the transformation from a to b in the form of integral-power series be a canonical one and derive the coefficients of the canonical transformation and the kernels of the five-wave reduced equation. Expansion of the Hamiltonian for capillary–gravity waves on a fluid of finite depth in integral-power series with accuracy up to the fifth-order terms is given, for completeness, in §4. A discussion and some comparisons with other approximations are given, finally, in §5.

2. Background and general considerations

Let $\zeta(\boldsymbol{x},t)$ be the vertical displacement of the surface of an inviscid laterally unbounded fluid of constant depth h above the point $\boldsymbol{x}=(x,y)$ at time t, $\phi(\boldsymbol{x},z,t)$ be the velocity potential for an irrotational flow moving under the influence of gravity (with g as gravitational acceleration) and surface tension (with γ as ratio of the surface tension coefficient to the fluid density), z be the vertical coordinate directed upwards with its origin on the undisturbed surface $z=0$, and $\psi(\boldsymbol{x},t)=\phi[\boldsymbol{x},\zeta(\boldsymbol{x},t),t]$ be the velocity potential evaluated on the surface. Then the evolution equation describing the wave motion can be put in the form of Hamilton canonical equations with the pair of canonically conjugate variables ζ and ψ (Zakharov 1968; Broer 1974; Miles 1977):

$$\frac{\partial\zeta(\boldsymbol{x},t)}{\partial t}=\frac{\delta H}{\delta\psi(\boldsymbol{x},t)},\quad \frac{\partial\psi(\boldsymbol{x},t)}{\partial t}=-\frac{\delta H}{\delta\zeta(\boldsymbol{x},t)},\tag{2.1}$$

where δ stands for functional derivatives, and the Hamiltonian (the total energy) $H=K+\Pi$ is the sum of the kinetic (K) and potential (Π) energies divided by the fluid density. These are given by

$$K=\frac{1}{2}\int\int_{-h}^{\zeta}\left[(\boldsymbol{\nabla}\phi)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]\mathrm{d}z\,\mathrm{d}\boldsymbol{x},\tag{2.2}$$

$$\Pi=\frac{1}{2}\int\zeta^2\,\mathrm{d}\boldsymbol{x}+\gamma\int\{[1+(\boldsymbol{\nabla}\zeta)^2]^{\frac{1}{2}}-1\}\,\mathrm{d}\boldsymbol{x},\tag{2.3}$$

where $\boldsymbol{\nabla}=(\partial/\partial x,\partial/\partial y)$ is the horizontal gradient operator and integration with respect to $\boldsymbol{x}$ is extended over the entire horizontal plane. The velocity potential must satisfy the Laplace equation $\nabla^2\phi+\partial^2\phi/\partial z^2=0$ in the domain $-\infty<x,y<+\infty$, $-h<z<\zeta(\boldsymbol{x},t)$ and the boundary conditions on the bottom $\partial\phi/\partial z=0$ for $z=-h$. The kinematical and dynamical boundary conditions at the surface are not required for the Laplace equation because they are taken into account by the above Hamiltonian formulation. In (2.1) the Hamiltonian should be considered as a functional of ζ and ψ.

Introduce the Fourier representations of $\zeta(\boldsymbol{x},t)$ and $\psi(\boldsymbol{x},t)$ by the relations

$$\zeta(\boldsymbol{x})=\frac{1}{2\pi}\int\zeta(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{k},\quad \zeta(\boldsymbol{k})=\zeta^*(-\boldsymbol{k}),\tag{2.4}$$

$$\psi(\boldsymbol{x})=\frac{1}{2\pi}\int\psi(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{k},\quad \psi(\boldsymbol{k})=\psi^*(-\boldsymbol{k}),\tag{2.5}$$

where $\boldsymbol{k}=(k_x,k_y)$ is the horizontal wave vector, integration with respect to $\boldsymbol{k}$ is extended over the entire $\boldsymbol{k}$-plane, the asterisk denotes complex conjugate, and explicit dependence of ζ and ψ on t is suppressed for simplicity of notation. We denote here functions and their Fourier transforms by the same symbols, distinguishing them by their arguments. The Fourier transformation is a canonical one and thus reduces the canonical equations (2.1) into the canonical ones

$$\frac{\partial\zeta(\boldsymbol{k})}{\partial t}=\frac{\delta H}{\delta\psi^*(\boldsymbol{k})},\quad \frac{\partial\psi(\boldsymbol{k})}{\partial t}=-\frac{\delta H}{\delta\zeta^*(\boldsymbol{k})},\tag{2.6}$$

with the pair of canonically conjugate variables $\zeta(\boldsymbol{k})$ and $\psi^*(\boldsymbol{k})$. Now H is a functional of $\zeta(\boldsymbol{k})$, $\zeta^*(\boldsymbol{k})$, $\psi(\boldsymbol{k})$, $\psi^*(\boldsymbol{k})$.

Further canonical transformation to the new pair of canonically conjugate variables $a(\boldsymbol{k})$ and $\mathrm{i}a^*(\boldsymbol{k})$ defined by the relations

$$\zeta(\boldsymbol{k}) = \mathscr{M}(\boldsymbol{k})[a(\boldsymbol{k})+a^*(-\boldsymbol{k})], \quad \psi(\boldsymbol{k}) = -\mathrm{i}\mathscr{N}(\boldsymbol{k})[a(\boldsymbol{k})-a^*(-\boldsymbol{k})], \tag{2.7}$$

with

$$\mathscr{M}(\boldsymbol{k}) = \left[\frac{q(\boldsymbol{k})}{2\omega(\boldsymbol{k})}\right]^{\frac{1}{2}}, \quad \mathscr{N}(\boldsymbol{k}) = \left[\frac{\omega(\boldsymbol{k})}{2q(\boldsymbol{k})}\right]^{\frac{1}{2}}, \tag{2.8}$$

where $\omega(\boldsymbol{k})$ is the dispersion relation of linear waves defined by

$$\omega(\boldsymbol{k}) = [\tau(\boldsymbol{k})\,q(\boldsymbol{k})]^{\frac{1}{2}}, \quad \tau(\boldsymbol{k}) = g+\gamma|\boldsymbol{k}|^2, \quad q(\boldsymbol{k}) = |\boldsymbol{k}|\tanh(|\boldsymbol{k}|h), \tag{2.9}$$

reduces (2.6) to the single equation

$$\mathrm{i}\frac{\partial a(\boldsymbol{k})}{\partial t} = \frac{\delta H}{\delta a^*(\boldsymbol{k})}. \tag{2.10}$$

Here H is a functional of $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$. Equation (2.10) and its complex-conjugate form the pair of canonical Hamilton equations.

We are interested in waves of small but finite amplitudes, i.e. in weakly nonlinear waves. Assuming small wave slopes, we can formally expand the Hamiltonian $H = H(a, a^*)$ into integral power series in powers of a and a^* (see §4). For weakly nonlinear waves one can retain in the expansion a finite number of terms. (Some effects of truncating the Hamiltonian were recently studied by Milder 1990.) Here we consider this expansion with accuracy up to and including the fifth-order terms. It is convenient to place this expansion in the form

$$\begin{aligned} H = &\int \omega_0\, a_0^*\, a_0\, \mathrm{d}\boldsymbol{k}_0 + \int U^{(1)}_{0,1,2}(a_0^*\, a_1\, a_2 + a_0\, a_1^*\, a_2^*)\,\delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{012} \\ &+\frac{1}{3}\int U^{(3)}_{0,1,2}(a_0^*\, a_1^*\, a_2^* + a_0\, a_1\, a_2)\,\delta_{0+1+2}\,\mathrm{d}\boldsymbol{k}_{012} \\ &+\int V^{(1)}_{0,1,2,3}(a_0^*\, a_1\, a_2\, a_3 + a_0\, a_1^*\, a_2^*\, a_3^*)\,\delta_{0-1-2-3}\,\mathrm{d}\boldsymbol{k}_{0123} \\ &+\frac{1}{2}\int V^{(2)}_{0,1,2,3}\, a_0^*\, a_1^*\, a_2\, a_3\,\delta_{0+1-2-3}\,\mathrm{d}\boldsymbol{k}_{0123} \\ &+\frac{1}{4}\int V^{(4)}_{0,1,2,3}(a_0^*\, a_1^*\, a_2^*\, a_3^* + a_0\, a_1\, a_2\, a_3)\,\delta_{0+1+2+3}\,\mathrm{d}\boldsymbol{k}_{0123} \\ &+\int W^{(1)}_{0,1,2,3,4}(a_0^*\, a_1\, a_2\, a_3\, a_4 + a_0\, a_1^*\, a_2^*\, a_3^*\, a_4^*)\,\delta_{0-1-2-3-4}\,\mathrm{d}\boldsymbol{k}_{01234} \\ &+\frac{1}{2}\int W^{(2)}_{0,1,2,3,4}(a_0^*\, a_1^*\, a_2\, a_3\, a_4 + a_0\, a_1\, a_2^*\, a_3^*\, a_4^*)\,\delta_{0+1-2-3-4}\,\mathrm{d}\boldsymbol{k}_{01234} \\ &+\frac{1}{5}\int W^{(5)}_{0,1,2,3,4}(a_0^*\, a_1^*\, a_2^*\, a_3^*\, a_4^* + a_0\, a_1\, a_2\, a_3\, a_4)\,\delta_{0+1+2+3+4}\,\mathrm{d}\boldsymbol{k}_{01234}, \end{aligned} \tag{2.11}$$

where the perturbation parameter (the wave slope) has been drawn into a.

In expansion (2.11) we have introduced the compact notation in which the arguments $\boldsymbol{k}_j$ in a, ω, $U^{(n)}$, $V^{(n)}$, $W^{(n)}$ and δ-functions are replaced by subscripts j, with the subscript zero assigned to $\boldsymbol{k}$. Thus, for example, $a_j = a(\boldsymbol{k}_j, t)$, $\omega_j = \omega(\boldsymbol{k}_j)$, $U^{(n)}_{0,1,2} = U^{(n)}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$, $\delta_{0-1-2} = \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)$, etc. For differentials we have used the notation $\mathrm{d}\boldsymbol{k}_0 = \mathrm{d}\boldsymbol{k}$, $\mathrm{d}\boldsymbol{k}_{012} = \mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2$, etc., and the integral signs denote corresponding multiple integrals with the limits from $-\infty$ to $+\infty$. The normalization coefficients (2.8) are defined so to present the quadratic part of the Hamiltonian as in (2.11). The nth order parts of the Hamiltonian describe n-wave interactions.

It is convenient to assume that the coefficients $U^{(n)}$, $V^{(n)}$, $W^{(n)}$ satisfy the 'natural symmetry conditions', which specify that the integrals in (2.11) are unaffected by relabelling of the dummy integration variables. Thus, the coefficient $U^{(1)}_{0,1,2}$ should be considered as symmetric under the transposition of the arguments 1 and 2, as are $U^{(3)}_{0,1,2}$ under all the transpositions of 0, 1 and 2, $V^{(1)}_{0,1,2,3}$ under all the transpositions 1, 2 and 3, $V^{(2)}_{0,1,2,3}$ under the transpositions of the arguments inside the groups (0, 1) and (2, 3), $W^{(2)}_{0,1,2,3,4}$ under all the transpositions of the arguments inside the groups (0, 1) and (2, 3, 4), and so on. The coefficients should also satisfy symmetry conditions expressing the reality of the Hamiltonian. There is only one such condition for the Hamiltonian in the form (2.11), namely the coefficient $V^{(2)}_{0,1,2,3}$ should be symmetric under the transpositions of the argument pairs (0, 1) and (2, 3), i.e. $V^{(2)}_{0,1,2,3} = V^{(2)}_{2,3,0,1}$. Thus, the coefficient $V^{(2)}_{0,1,2,3}$ should satisfy the following symmetry conditions:

$$V^{(2)}_{0,1,2,3} = V^{(2)}_{1,0,2,3} = V^{(2)}_{0,1,3,2} = V^{(2)}_{2,3,0,1}. \tag{2.12}$$

We note that when calculating the coefficients using the Hamiltonian (see §4) they, as a rule, do not satisfy natural symmetry conditions and, therefore, should be symmetrized by substituting for the sums of non-symmetrical coefficients, taken over all transpositions of corresponding groups of arguments, divided by the number of these transpositions. The symmetry expressing reality of the Hamiltonian must result automatically.

By virtue of (2.10), the following evolution equation corresponds to the Hamiltonian (2.11):

$$\begin{aligned}
\mathrm{i}\frac{\partial a_0}{\partial t} = \frac{\delta H}{\delta a_0^*} = \omega_0 a_0 &+ \int U^{(1)}_{0,1,2}\, a_1 a_2\, \delta_{0-1-2}\, \mathrm{d}\boldsymbol{k}_{12} \\
&+ 2\int U^{(1)}_{2,1,0}\, a_1^* a_2\, \delta_{0+1-2}\, \mathrm{d}\boldsymbol{k}_{12} + \int U^{(3)}_{0,1,2}\, a_1^* a_2^*\, \delta_{0+1+2}\, \mathrm{d}\boldsymbol{k}_{12} \\
&+ \int V^{(1)}_{0,1,2,3}\, a_1 a_2 a_3\, \delta_{0-1-2-3}\, \mathrm{d}\boldsymbol{k}_{123} \\
&+ \int V^{(2)}_{0,1,2,3}\, a_1^* a_2 a_3\, \delta_{0+1-2-3}\, \mathrm{d}\boldsymbol{k}_{123} \\
&+ 3\int V^{(1)}_{3,2,1,0}\, a_1^* a_2^* a_3\, \delta_{0+1+2-3}\, \mathrm{d}\boldsymbol{k}_{123} \\
&+ \int V^{(4)}_{0,1,2,3}\, a_1^* a_2^* a_3^*\, \delta_{0+1+2+3}\, \mathrm{d}\boldsymbol{k}_{123} \\
&+ \int W^{(1)}_{0,1,2,3,4}\, a_1 a_2 a_3 a_4\, \delta_{0-1-2-3-4}\, \mathrm{d}\boldsymbol{k}_{1234} \\
&+ \int W^{(2)}_{0,1,2,3,4}\, a_1^* a_2 a_3 a_4\, \delta_{0+1-2-3-4}\, \mathrm{d}\boldsymbol{k}_{1234}
\end{aligned}$$

$$+\frac{3}{2}\int W^{(2)}_{4,3,2,1,0}\,a_1^*\,a_2^*\,a_3\,a_4\,\delta_{0+1+2-3-4}\,\mathrm{d}k_{1234}$$

$$+4\int W^{(1)}_{4,3,2,1,0}\,a_1^*\,a_2^*\,a_3^*\,a_4\,\delta_{0+1+2+3-4}\,\mathrm{d}k_{1234}$$

$$+\int W^{(5)}_{0,1,2,3,4}\,a_1^*\,a_2^*\,a_3^*\,a_4^*\,\delta_{0+1+2+3+4}\,\mathrm{d}k_{1234}. \tag{2.13}$$

The Hamiltonian structure of this integrodifferential equation is expressed, in particular, in the fact that not all individual integrals on the right-hand side have different kernels – some of them have the same upper indices. When deriving an analogous equation from the original hydrodynamical equations of surface wave theory this Hamiltonian structure is not observed and all the kernels turn out to be different (Yuen & Lake 1982; Stiassnie & Shemer 1984). Certainly, after proper symmetrization of the kernels, obtained from the original hydrodynamical equations, both methods of derivation of evolution equation for a must lead to the same result, but the symmetry conditions are not clear without the Hamiltonian formulation.

Consider now a canonical transformation from variable $a(\boldsymbol{k})$ to a new variable $b(\boldsymbol{k})$. A transformation $a = a(b, b^*)$ will be canonical if the evolution equation for $b(\boldsymbol{k})$ has the same Hamiltonian form as (2.10), i.e.

$$\mathrm{i}\frac{\partial b(\boldsymbol{k})}{\partial t} = \frac{\delta \tilde{H}}{\delta b^*(\boldsymbol{k})}, \tag{2.14}$$

where $\tilde{H} = \tilde{H}(b, b^*)$ is the Hamiltonian $H = H(a, a^*)$ after substitution of the transformation $a = a(b, b^*)$.

For the transformation to be a canonical one some conditions, which we will term 'canonicity conditions', must be satisfied. There are different forms of canonicity conditions in classical mechanics. They are easily generalized to the continuous case. One of them, probably the most well known, is expressed through the Poisson brackets:

$$\int\left[\frac{\delta a(\boldsymbol{k})}{\delta b(\boldsymbol{q})}\frac{\delta a(\boldsymbol{k}')}{\delta b^*(\boldsymbol{q})} - \frac{\delta a(\boldsymbol{k})}{\delta b^*(\boldsymbol{q})}\frac{\delta a(\boldsymbol{k}')}{\delta b(\boldsymbol{q})}\right]\mathrm{d}\boldsymbol{q} = 0, \tag{2.15}$$

$$\int\left[\frac{\delta a(\boldsymbol{k})}{\delta b(\boldsymbol{q})}\frac{\delta a^*(\boldsymbol{k}')}{\delta b^*(\boldsymbol{q})} - \frac{\delta a(\boldsymbol{k})}{\delta b^*(\boldsymbol{q})}\frac{\delta a^*(\boldsymbol{k}')}{\delta b(\boldsymbol{q})}\right]\mathrm{d}\boldsymbol{q} = \delta(\boldsymbol{k}-\boldsymbol{k}'). \tag{2.16}$$

Postulate the canonical transformation from $a(\boldsymbol{k})$ to $b(\boldsymbol{k})$ in the form of integral-power series:

$$\begin{aligned} a_0 = b_0 &+ \int A^{(1)}_{0,1,2}\,b_1\,b_2\,\delta_{0-1-2}\,\mathrm{d}k_{12} + \int A^{(2)}_{0,1,2}\,b_1^*\,b_2\,\delta_{0+1-2}\,\mathrm{d}k_{12} \\ &+ \int A^{(3)}_{0,1,2}\,b_1^*\,b_2^*\,\delta_{0+1+2}\,\mathrm{d}k_{12} + \int B^{(1)}_{0,1,2,3}\,b_1\,b_2\,b_3\,\delta_{0-1-2-3}\,\mathrm{d}k_{123} \\ &+ \int B^{(2)}_{0,1,2,3}\,b_1^*\,b_2\,b_3\,\delta_{0+1-2-3}\,\mathrm{d}k_{123} + \int B^{(3)}_{0,1,2,3}\,b_1^*\,b_2^*\,b_3\,\delta_{0+1+2-3}\,\mathrm{d}k_{123} \\ &+ \int B^{(4)}_{0,1,2,3}\,b_1^*\,b_2^*\,b_3^*\,\delta_{0+1+2+3}\,\mathrm{d}k_{123} + \int C^{(1)}_{0,1,2,3,4}\,b_1\,b_2\,b_3\,b_4\,\delta_{0-1-2-3-4}\,\mathrm{d}k_{1234} \\ &+ \int C^{(2)}_{0,1,2,3,4}\,b_1^*\,b_2\,b_3\,b_4\,\delta_{0+1-2-3-4}\,\mathrm{d}k_{1234} \end{aligned}$$

$$+\int C^{(3)}_{0,1,2,3,4}\, b_1^* b_2^* b_3 b_4\, \delta_{0+1+2-3-4}\, \mathrm{d}k_{1234}$$

$$+\int C^{(4)}_{0,1,2,3,4}\, b_1^* b_2^* b_3^* b_4\, \delta_{0+1+2+3-4}\, \mathrm{d}k_{1234}$$

$$+\int C^{(5)}_{0,1,2,3,4}\, b_1^* b_2^* b_3^* b_4^*\, \delta_{0+1+2+3+4}\, \mathrm{d}k_{1234}+\dots. \tag{2.17}$$

In the following, we will suppose that the coefficients $A^{(n)}$, $B^{(n)}$, $C^{(n)}$ satisfy proper natural symmetry conditions. The canonicity conditions (2.15) and (2.16) impose some constraints on the coefficients $A^{(n)}$, $B^{(n)}$, $C^{(n)}$ which will be briefly discussed in the next paragraph.

The Hamiltonian $\tilde{H}$ is obtained by substitution of (2.17) into (2.11) and, with accuracy up to the fifth-order terms, has the same form as (2.11) but replacing a_j by b_j, and $U^{(n)}$, $V^{(n)}$, $W^{(n)}$ by new coefficients $\tilde{U}^{(n)}$, $\tilde{V}^{(n)}$, $\tilde{W}^{(n)}$ resulting from the substitution indicated. Obviously, these new coefficients must satisfy the same symmetry conditions as the old ones. In particular

$$\tilde{V}^{(2)}_{0,1,2,3} = \tilde{V}^{(2)}_{1,0,2,3} = \tilde{V}^{(2)}_{0,1,3,2} = \tilde{V}^{(2)}_{2,3,0,1}. \tag{2.18}$$

The canonical transformation enables a fundamental simplification (or, in other words, reduction) of the Hamiltonian $\tilde{H}$ eliminating therein 'unimportant' terms by suitable choice of the coefficients $A^{(n)}$, $B^{(n)}$, $C^{(n)}$. The reduction of the Hamiltonian is crucially dependent on the shape of the dispersion curve $\omega(\boldsymbol{k})$, being different for capillary–gravity (or purely capillary) and purely gravity waves. Consider these cases separately.

Capillary–gravity waves: three-wave interactions. Consider first, for simplicity, the case $V^{(n)} = W^{(n)} = 0$ and put

$$A^{(1)}_{0,1,2} = A^{(2)}_{0,1,2} = 0, \quad A^{(3)}_{0,1,2} = -\frac{U^{(3)}_{0,1,2}}{\omega_0+\omega_1+\omega_2}, \quad B^{(n)} = C^{(n)} = 0. \tag{2.19}$$

This transformation is canonical up to the second-order terms in b and gives $\tilde{U}^{(1)} = U^{(1)}$, $\tilde{U}^{(3)} = 0$. Thus, in this case, the reduced Hamiltonian is

$$\tilde{H} = \int \omega_0 b_0^* b_0\, \mathrm{d}k_0 + \int U^{(1)}_{0,1,2}(b_0^* b_1 b_2 + b_0 b_1^* b_2^*)\,\delta_{0-1-2}\,\mathrm{d}k_{012}. \tag{2.20}$$

Accordingly to (2.14) the corresponding *three-wave reduced equation* is

$$\mathrm{i}\frac{\partial b_0}{\partial t} = \frac{\delta\tilde{H}}{\delta b_0^*} = \omega_0 b_0 + \int U^{(1)}_{0,1,2} b_1 b_2\, \delta_{0-1-2}\,\mathrm{d}k_{12} + 2\int U^{(1)}_{2,1,0} b_1^* b_2\, \delta_{0+1-2}\,\mathrm{d}k_{12}. \tag{2.21}$$

Purely gravity waves: four-wave interactions. Put $C^{(n)} = 0$, $W^{(n)} = 0$. In this case the coefficients $A^{(n)}$ and $B^{(n)}$ can be chosen so to make $\tilde{U}^{(m)} = 0$, $m = 1,3$ and $\tilde{V}^{(n)} = 0$, $n = 1,4$. Thus, the reduced Hamiltonian is

$$\tilde{H} = \int \omega_0 b_0^* b_0\,\mathrm{d}k_0 + \frac{1}{2}\int \tilde{V}^{(2)}_{0,1,2,3} b_0^* b_1^* b_2 b_3\, \delta_{0+1-2-3}\,\mathrm{d}k_{0123} \tag{2.22}$$

and the corresponding *four-wave reduced equation* reads

$$\mathrm{i}\frac{\partial b_0}{\partial t} = \frac{\delta\tilde{H}}{\delta b_0^*} = \omega_0 b_0 + \int \tilde{V}^{(2)}_{0,1,2,3} b_1^* b_2 b_3\, \delta_{0+1-2-3}\,\mathrm{d}k_{123}. \tag{2.23}$$

The coefficient $\tilde{V}^{(2)}$ is given in explicit form in §3.

Purely gravity waves: combined four- and five-wave interactions. By a suitable choice of the coefficients $A^{(n)}$, $B^{(n)}$, $C^{(n)}$ in the full form of the canonical transformation (2.17) one can get $\tilde{U}^{(m)}=0$, $m=1,3$, $\tilde{V}^{(n)}=0$, $n=1,4$, $\tilde{W}^{(p)}=0$, $p=1,5$, so the reduced Hamiltonian takes the form

$$\tilde{H}=\int \omega_0 b_0^* b_0 \,\mathrm{d}k_0+\frac{1}{2}\int \tilde{V}^{(2)}_{0,1,2,3} b_0^* b_1^* b_2 b_3 \delta_{0+1-2-3}\,\mathrm{d}k_{0123}$$
$$+\frac{1}{2}\int \tilde{W}^{(2)}_{0,1,2,3,4}(b_0^* b_1^* b_2 b_3 b_4+b_0 b_1 b_2^* b_3^* b_4^*)\,\delta_{0+1-2-3-4}\,\mathrm{d}k_{01234}, \tag{2.24}$$

and the corresponding *five-wave reduced equation* becomes

$$\mathrm{i}\frac{\partial b_0}{\partial t}=\frac{\delta \tilde{H}}{\delta b_0^*}=\omega_0 b_0+\int \tilde{V}^{(2)}_{0,1,2,3} b_1^* b_2 b_3 \delta_{0+1-2-3}\,\mathrm{d}k_{123}$$
$$+\int \tilde{W}^{(2)}_{0,1,2,3,4} b_1^* b_2 b_3 b_4 \delta_{0+1-2-3-4}\,\mathrm{d}k_{1234}$$
$$+\frac{3}{2}\int \tilde{W}^{(2)}_{4,3,2,1,0} b_1^* b_2^* b_3 b_4 \delta_{0+1+2-3-4}\,\mathrm{d}k_{1234}. \tag{2.25}$$

The coefficient $\tilde{W}^{(2)}$ is also given in explicit form in §3.

In reducing the Hamiltonian H we come up against the 'problem of small divisors', formally resembling the situation in celestial mechanics and related in the present case to the appearance of non-integrable singularities in the coefficients of the canonical transformation near the manifolds

$$\Delta_{0-1-2}=0, \quad \boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2=0, \tag{2.26}$$
$$\Delta_{0+1-2-3}=0, \quad \boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3=0, \tag{2.27}$$
$$\Delta_{0+1-2-3-4}=0, \quad \boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4=0, \tag{2.28}$$
$$\Delta_{0+1+2-3-4}=0, \quad \boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4=0, \tag{2.29}$$

where notation like $\Delta_{0-1-2}=\omega_0-\omega_1-\omega_2$, etc. is introduced. (Equations (2.26)–(2.29) are termed the resonance conditions, and the frequency differences involved, Δ, the resonant ones. Note, that the resonance conditions (2.28) and (2.29) are equivalent in the sense that the former follows from the latter by renumbering of wave vectors.) For instance, the canonical transformation (2.19) is possible because for capillary–gravity waves the conditions $\Delta_{0+1+2}=0$, $\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2=0$ cannot be satisfied. In the case of capillary–gravity waves, an attempt to eliminate the term with $\tilde{U}^{(1)}$ from the Hamiltonian $\tilde{H}$ leads to singularities in $A^{(1)}$ and $A^{(2)}$ near the manifold (2.26). In the case of purely gravity waves, the conditions (2.26) cannot be satisfied and the cubic part of $\tilde{H}$ (i.e. the terms with $\tilde{U}^{(1)}$ and $\tilde{U}^{(3)}$) can be completely eliminated. But an attempt to eliminate the terms with $\tilde{V}^{(2)}$ and $\tilde{W}^{(2)}$ leads to singularities in $B^{(2)}$, $C^{(2)}$ and $C^{(3)}$ near the manifolds (2.27)–(2.29), correspondingly (these statements will be more clear from the results of §3). The non-resonant terms in the Hamiltonian $\tilde{H}$, which can be eliminated by suitable canonical transformations, are, in a sense, unimportant.

All of the above reduced Hamiltonians are obvious integrals of motion, i.e. they all conserve the total energy. In addition, there are other integrals of motion (see §3).

Note that instead of (2.23) one often uses the equation

$$\mathrm{i}\frac{\partial B_0}{\partial t}=\int \tilde{V}^{(2)}_{0,1,2,3} B_1^* B_2 B_3 \exp[\mathrm{i}\Delta_{0+1-2-3}t]\,\delta_{0+1-2-3}\,\mathrm{d}k_{123},$$

which is obtained from (2.23) after the change of variable

$$b(\boldsymbol{k}, t) = B(\boldsymbol{k}, t)\exp[-\mathrm{i}\omega(\boldsymbol{k})\,t].$$

This equation describes the slow evolution of a gravity wave field due to weakly nonlinear four-wave interactions. The non-Hamiltonian version of this equation was first derived by Zakharov (1966, 1968) from equation (2.13), with $V^{(m)} = 0$, $m = 1, 4$ and $W^{(n)} = 0$, $n = 1, 2, 5$, by heuristic considerations (see, also, Crawford *et al.* 1980). The method of derivation suggests that the slow evolution of $B(\boldsymbol{k}, t)$ is determined mainly by interactions of wave fours approximately satisfying the resonance conditions (2.27). This method gives the reduced equation whose kernel does not satisfy all symmetry conditions (2.18) (it possesses only the symmetry relative to transpositions of the arguments 2 and 3), and thus is not Hamiltonian. The canonical transformation technique, described in detail below, automatically removes this shortcoming.

3. The coefficients of canonical transformation and the kernels of reduced equations

In the following paragraphs, attention is paid primarily to the five-wave reduced equation (2.25), which, in a sense, comprises the four-wave equation (2.23), so the latter, generally speaking, should not be derived separately. Derivation of (2.23) by the procedure outlined above of direct reducing the Hamiltonian to the form (2.22) taking account of the canonicity conditions (2.15) and (2.16) is given in some detail in Krasitskii (1990), but without reference to water wave problems (for a concise discussion, more close to the present paper, see also Krasitskii 1991). But this procedure, applied for deriving (2.24), leads not only to extremely cumbersome algebra, but also to some difficulties of taking account of canonicity conditions in the form (2.15) and (2.16), which are rather complicated as well. An equivalent method of directly reducing the Hamilton equation (2.13) to the Hamilton reduced equation (2.25) using the transformation (2.17) turns out to be far more simple. It is the method which we will use below.

To realize the method one should substitute the transformation (2.17) into (2.13), substitute arising derivatives of $\partial b_j/\partial t$ from equation (2.25) and then collect, after proper symmetrization, the kernels of the integrals with the factors $b_1 b_2$, $b_1^* b_2, \ldots$, $b_1^* b_2^* b_3^* b_4^*$. As a result, we obtain the following twelve equations:

$$U^{(1)}_{0,1,2} + \Delta_{0-1-2} A^{(1)}_{0,1,2} = 0, \tag{3.1}$$

$$2U^{(1)}_{2,1,0} + \Delta_{0+1-2} A^{(2)}_{0,1,2} = 0, \tag{3.2}$$

$$U^{(3)}_{0,1,2} + \Delta_{0+1+2} A^{(3)}_{0,1,2} = 0, \tag{3.3}$$

$$Z^{(1)}_{0,1,2,3} + V^{(1)}_{0,1,2,3} + \Delta_{0-1-2-3} B^{(1)}_{0,1,2,3} = 0, \tag{3.4}$$

$$\tilde{V}^{(2)}_{0,1,2,3} = Z^{(2)}_{0,1,2,3} + V^{(2)}_{0,1,2,3} + \Delta_{0+1-2-3} B^{(2)}_{0,1,2,3}, \tag{3.5}$$

$$Z^{(3)}_{0,1,2,3} + 3V^{(1)}_{3,2,1,0} + \Delta_{0+1+2-3} B^{(3)}_{0,1,2,3} = 0, \tag{3.6}$$

$$Z^{(4)}_{0,1,2,3} + V^{(4)}_{0,1,2,3} + \Delta_{0+1+2+3} B^{(4)}_{0,1,2,3} = 0, \tag{3.7}$$

$$X^{(1)}_{0,1,2,3,4} + W^{(1)}_{0,1,2,3,4} + \Delta_{0-1-2-3-4} C^{(1)}_{0,1,2,3,4} = 0, \tag{3.8}$$

$$\tilde{W}^{(2)}_{0,1,2,3,4} = X^{(2)}_{0,1,2,3,4} + W^{(2)}_{0,1,2,3,4} + \Delta_{0+1-2-3-4} C^{(2)}_{0,1,2,3,4}, \tag{3.9}$$

$$\tfrac{3}{2}\tilde{W}^{(2)}_{4,3,2,1,0} = X^{(3)}_{0,1,2,3,4} + \tfrac{3}{2}W^{(2)}_{4,3,2,1,0} + \Delta_{0+1+2-3-4} C^{(3)}_{0,1,2,3,4}, \tag{3.10}$$

$$X^{(4)}_{0,1,2,3,4} + 4W^{(1)}_{4,3,2,1,0} + \Delta_{0+1+2+3-4} C^{(4)}_{0,1,2,3,4} = 0, \tag{3.11}$$

$$X^{(5)}_{0,1,2,3,4} + W^{(5)}_{0,1,2,3,4} + \Delta_{0+1+2+3+4} C^{(5)}_{0,1,2,3,4} = 0. \tag{3.12}$$

The functions $Z^{(n)}$ and $X^{(n)}$ are given in an Appendix.† For given n the symmetry properties of the functions $Z^{(n)}$ and $X^{(n)}$ are the same as for the coefficients $B^{(n)}$ and $C^{(n)}$ correspondingly.

The coefficients $A^{(n)}$, $B^{(n)}$, $C^{(n)}$ can be subdivided into the two groups: 'non-resonant' ($A^{(1)}, A^{(2)}, A^{(3)}, B^{(1)}, B^{(3)}, B^{(4)}, C^{(1)}, C^{(4)}, C^{(5)}$) and 'resonant' ($B^{(2)}, C^{(2)}, C^{(3)}$) ones. The non-resonant coefficients are derived immediately from the above equations:

$$A^{(1)}_{0,1,2} = -\Delta^{-1}_{0-1-2}\, U^{(1)}_{0,1,2}, \tag{3.13}$$

$$A^{(2)}_{0,1,2} = -2\Delta^{-1}_{0+1-2}\, U^{(1)}_{2,1,0} = -2A^{(1)}_{2,1,0}, \tag{3.14}$$

$$A^{(3)}_{0,1,2} = -\Delta^{-1}_{0+1+2}\, U^{(3)}_{0,1,2}, \tag{3.15}$$

$$B^{(1)}_{0,1,2,3} = -\Delta^{-1}_{0-1-2-3}[Z^{(1)}_{0,1,2,3} + V^{(1)}_{0,1,2,3}], \tag{3.16}$$

$$B^{(3)}_{0,1,2,3} = -\Delta^{-1}_{0+1+2-3}[Z^{(3)}_{0,1,2,3} + 3V^{(1)}_{3,2,1,0}], \tag{3.17}$$

$$B^{(4)}_{0,1,2,3} = -\Delta^{-1}_{0+1+2+3}[Z^{(4)}_{0,1,2,3} + V^{(4)}_{0,1,2,3}], \tag{3.18}$$

$$C^{(1)}_{0,1,2,3,4} = -\Delta^{-1}_{0-1-2-3-4}[X^{(1)}_{0,1,2,3,4} + W^{(1)}_{0,1,2,3,4}], \tag{3.19}$$

$$C^{(4)}_{0,1,2,3,4} = -\Delta^{-1}_{0+1+2+3-4}[X^{(4)}_{0,1,2,3,4} + 4W^{(1)}_{4,3,2,1,0}], \tag{3.20}$$

$$C^{(5)}_{0,1,2,3,4} = -\Delta^{-1}_{0+1+2+3+4}[X^{(5)}_{0,1,2,3,4} + W^{(5)}_{0,1,2,3,4}]. \tag{3.21}$$

All non-resonant coefficients are inversely proportional to 'the non-resonant frequency differences' which cannot vanish for a purely gravity wave dispersion law. For example, for the coefficient $B^{(1)}$ this means that the system of equations $\Delta_{0-1-2-3} = 0$, $\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3 = 0$ has no solutions. This is the circumstance which allows the possibility of reducing the Hamiltonian H, i.e. elimination of the non-resonant ('unimportant') terms from it. An attempt to eliminate the resonant terms using the conditions $\tilde{V}^{(2)} = 0$, $\tilde{W}^{(2)} = 0$, in (3.5), (3.9), (3.10) leads to the appearance of non-integrable singularities in the coefficients $B^{(2)}$, $C^{(2)}$, $C^{(3)}$ near the manifolds defined, respectively, by (2.27)–(2.29), as was mentioned in §2. Note that relations (3.13)–(3.18) were derived by Krasitskii (1990) through the canonicity conditions obtained from the Poisson brackets and directly reducing the Hamiltonian H to the form (2.22), but by far more cumbersome calculations.

The resonant coefficients of the canonical transformation $B^{(2)}$, $C^{(2)}$, $C^{(3)}$ and the kernels of the reduced equation $\tilde{V}^{(2)}$, $\tilde{W}^{(2)}$ cannot be obtained uniquely from (3.5), (3.9), (3.10). It will be shown below that they can be determined correct to an arbitrary function satisfying some symmetry conditions. This is because the canonical transformation admits a certain freedom which for non-resonant coefficients is limited by the condition of exclusion of the non-resonant terms from H.

For determining $B^{(2)}$ and $\tilde{V}^{(2)}$ we turn to (3.5). Using the symmetry properties (2.12) and (2.18) for the kernels $V^{(2)}$ and $\tilde{V}^{(2)}$, it is easy to eliminate them from (3.5) and to obtain for $B^{(2)}$ the following equation:

$$\Delta_{0+1-2-3}[B^{(2)}_{0,1,2,3} + B^{(2)}_{3,2,1,0}] + Z^{(2)}_{0,1,2,3} - Z^{(2)}_{3,2,1,0} = 0, \tag{3.22}$$

which, actually, is the canonicity condition for $B^{(2)}$. It is clear from the structure of this equation that its general solution $B^{(2)}_{0,1,2,3} = B^{(2)}_{0,1,3,2}$ should be presented as a sum of some particular solution $\Lambda_{0,1,2,3} = \Lambda_{0,1,3,2}$ and an 'arbitrary function' λ satisfying the conditions $\lambda_{0,1,2,3} = \lambda_{0,1,3,2} = -\lambda_{3,2,1,0}$. The function λ can be chosen for convenience (changing $B^{(2)}$ simultaneously changes both $\tilde{V}^{(2)}$ and $b(\boldsymbol{k})$, but leaves $a(\boldsymbol{k})$ unchanged in the canonical transformation). In what follows, we will consider that the function λ is identically equal to zero, and $B^{(2)}_{0,1,2,3}$ is a suitable particular solution of the equation

† A copy of the Appendix is available from the Editorial Office or the author.

(3.22). This particular solution should be symmetric under the transpositions of the arguments 2 and 3, and non-singular when $\Delta_{0+1-2-3} \rightarrow 0$.

The structure of (3.22) suggests that this particular solution can be sought as a linear combination of the functions $Z^{(2)}$ of different combinations of indices divided by $\Delta_{0+1-2-3}$. This solution is easily constructed by elementary considerations and has the form

$$B^{(2)}_{0,1,2,3} = -\tfrac{1}{4}\Delta^{-1}_{0+1-2-3}[3Z^{(2)}_{0,1,2,3} - Z^{(2)}_{1,0,2,3} - Z^{(2)}_{2,3,0,1} - Z^{(2)}_{3,2,0,1}]. \tag{3.23}$$

This solution is, formally, singular when $\Delta_{0+1-2-3} \rightarrow 0$, but substituting in (3.23) the expression for $Z^{(2)}$ from the Appendix (see footnote on p. 000) and using (3.1) and (3.3) shows that $\Delta_{0+1-2-3}$ is just cancelled here yielding

$$\begin{aligned} B^{(2)}_{0,1,2,3} = {} & A^{(3)}_{0,1,-0-1} A^{(3)}_{2,3,-2-3} + A^{(1)}_{1,2,1-2} A^{(1)}_{3,0,3-0} + A^{(1)}_{1,3,1-3} A^{(1)}_{2,0,2-0} \\ & - A^{(1)}_{0+1,0,1} A^{(1)}_{2+3,2,3} - A^{(1)}_{0,2,0-2} A^{(1)}_{3,1,3-1} - A^{(1)}_{0,3,0-3} A^{(1)}_{2,1,2-1}. \end{aligned} \tag{3.24}$$

In Krasitskii (1990) the solution (3.24) was guessed as a solution of canonicity conditions derived through the Poisson brackets. Here it is found by a more systematic and simple way, close to that used below for determining the coefficient $C^{(2)}$. If the coefficient $B^{(2)}$ is known, the kernal $\tilde{V}^{(2)}$ can be found from (3.5), the right-hand side of which satisfies all the symmetry conditions in (2.18), but in implicit form (in (3.5) the symmetry described by (2.18) is exhibited by the total sum on the right-hand side, but not by each separate term). It is such representation of the $\tilde{V}^{(2)}$ with the symmetry properties in implicit form that is obtained when substituting (3.24) in (3.5).

Another representation for the kernel $\tilde{V}^{(2)}$ is obtained when substituting (3.23) in (3.5):

$$\tilde{V}^{(2)}_{0,1,2,3} = \tfrac{1}{4}[Z^{(2)}_{0,1,2,3} + Z^{(2)}_{1,0,2,3} + Z^{(2)}_{2,3,0,1} + Z^{(2)}_{3,2,0,1}] + V^{(2)}_{0,1,2,3}. \tag{3.25}$$

As distinct from (3.5), this expression depends on $\Delta_{0+1-2-3}$ in implicit form but, on the other hand, possesses all the symmetry properties in (2.18) in explicit form.

Note, that earlier (Zakharov 1966, 1968, 1974; Yuen & Lake 1982) the kernel $\tilde{V}^{(2)}_{0,1,2,3} = Z^{(2)}_{0,1,2,3} + V^{(2)}_{0,1,2,3}$ was used as the kernel of the four-wave reduced equation, which is obtained from (3.5) if one formally put $\Delta_{0+1-2-3} = 0$ therein. This kernel no longer satisfies all the symmetry conditions in (2.18): it is symmetric only with respect to transpositions of the arguments 2 and 3, but not for 0, 1 and for transpositions of the pairs (0, 1) and (2, 3). This violation of the symmetry properties of the kernel leads to violation of the Hamiltonian structure of equation (2.23) (see the more detailed discussion in Krasitskii 1990).

Now we turn to the determination of the canonical transformation coefficients $C^{(2)}$, $C^{(3)}$ and the kernel of the five-wave reduced equation $\tilde{W}^{(2)}$. Using the symmetry of the kernel $\tilde{W}^{(2)}_{0,1,2,3,3}$ under the transpositions of the arguments 0 and 1, it is easy to derive from (3.9) the following equation for $C^{(2)}$:

$$\Delta_{0+1-2-3-4}[C^{(2)}_{0,1,2,3,4} - C^{(2)}_{1,0,2,3,4}] + X^{(2)}_{0,1,2,3,4} - X^{(2)}_{1,0,2,3,4} = 0, \tag{3.26}$$

which is, essentially, the canonicity condition for $C^{(2)}$. It is clear that its general solution $C^{(2)}_{0,1,2,3,4}$, symmetric under the transpositions of the arguments 2, 3, 4, should be presented in the form of a sum of some particular solution $\Lambda_{0,1,2,3,4}$, symmetric under the transpositions of the arguments 2, 3, 4, and 'an arbitrary function' $\lambda_{0,1,2,3,4}$, symmetric under the transposition of the arguments inside the groups (0, 1) and (2, 3, 4). Assuming $\lambda \equiv 0$, we will consider $C^{(2)}_{0,1,2,3,4}$ as a suitable particular solution of (3.26). This particular solution should be symmetric under the transpositions of 2, 3, 4 and non-singular when $\Delta_{0+1-2-3-4} \rightarrow 0$. As such a particular solution one may take

$$C^{(2)}_{0,1,2,3,4} = -\tfrac{1}{2}\Delta^{-1}_{0+1-2-3-4}[X^{(2)}_{0,1,2,3,4} - X^{(2)}_{1,0,2,3,4}]. \tag{3.27}$$

Actually, the symmetry of this solution under 2, 3, 4 is evident. Besides, lengthy and cumbrous algebra shows that

$$X^{(2)}_{0,1,2,3,4}-X^{(2)}_{1,0,2,3,4}=-2\Delta_{0+1-2-3-4}[P_{0,1,2,3,4}-P_{1,0,2,3,4}], \tag{3.28}$$

where the function $P_{0,1,2,3,4}$ is symmetric under the transpositions of 2, 3, 4 and is presented in the form

$$P_{0,1,2,3,4}=p_{0,1,2,3,4}+p_{0,1,3,2,4}+p_{0,1,4,2,3}, \tag{3.29}$$

and the function $p_{0,1,2,3,4}$ is symmetric under 3, 4; it is given in the Appendix (see footnote on p. 10). Thus, when substituting (3.28) in (3.27), the resonant frequency difference $\Delta_{0+1-2-3-4}$ is cancelled yielding

$$C^{(2)}_{0,1,2,3,4}=P_{0,1,2,3,4}-P_{1,0,3,2,4}. \tag{3.30}$$

Note that this expression is antisymmetric under 0, 1.

The kernel $\tilde{W}^{(2)}$ is obtained by substituting (3.30) in (3.9). The resulting expression for $\tilde{W}^{(2)}_{0,1,2,3,4}$ possesses the necessary symmetry under the transpositions of the arguments inside the groups (0, 1) and (2, 3, 4) in implicit form. On the other hand, using nothing but the antisymmetry of $C^{(2)}_{0,1,2,3,4}$ under 0, 1, we find from (3.9)

$$\tilde{W}^{(2)}_{0,1,2,3,4}=\tfrac{1}{2}[X^{(2)}_{0,1,2,3,4}+X^{(2)}_{1,0,2,3,4}]+W^{(2)}_{0,1,2,3,4}. \tag{3.31}$$

This representation of the kernel already possesses all necessary symmetry properties in explicit form.

It remains to determine the coefficient $C^{(3)}_{0,1,2,3,4}$. It must be symmetric under the transpositions of the arguments inside the groups (1, 2) and (3, 4) and non-singular when $\Delta_{0+1+2-3-4}\to 0$. Eliminating $W^{(2)}$ and $\tilde{W}^{(2)}$ from (3.9) and (3.10), we obtain

$$C^{(3)}_{4,3,2,1,0}=\Delta^{-1}_{0+1-2-3-4}[X^{(3)}_{4,3,2,1,0}-\tfrac{3}{2}X^{(2)}_{0,1,2,3,4}]-\tfrac{3}{2}C^{(2)}_{0,1,2,3,4}. \tag{3.32}$$

Replacing the arguments 0 and 1 and adding the equality obtained to the initial one, we find

$$\begin{aligned}C^{(3)}_{4,3,2,1,0}&=\Delta^{-1}_{0+1-2-3-4}\{X^{(3)}_{4,3,2,1,0}-\tfrac{3}{4}[X^{(2)}_{0,1,2,3,4}+X^{(2)}_{1,0,2,3,4}]\}\\&=\tfrac{1}{2}\Delta^{-1}_{0+1-2-3-4}\{[X^{(3)}_{4,3,2,1,0}-\tfrac{3}{2}X^{(2)}_{0,1,2,3,4}]+[X^{(3)}_{4,3,2,0,1}-\tfrac{3}{2}X^{(2)}_{1,0,2,3,4}]\}.\end{aligned} \tag{3.33}$$

In contrast with (3.32), this expression is already symmetric under 0, 1 in explicit form. Lengthy algebra gives

$$X^{(3)}_{4,3,2,1,0}-\tfrac{3}{2}X^{(2)}_{0,1,2,3,4}=\Delta_{0+1-2-3-4}[Q_{4,3,2,1,0}+Q_{4,2,3,1,0}]. \tag{3.34}$$

The awkward function Q is given in the Appendix (see footnote on p. 10). Substituting (3.34) into (3.33) gives

$$C^{(3)}_{0,1,2,3,4}=\tfrac{1}{2}[Q_{0,1,2,3,4}+Q_{0,2,1,3,4}+Q_{0,1,2,4,3}+Q_{0,2,1,4,3}]. \tag{3.35}$$

Thus, all the coefficients of the canonical transformation and the kernels of the reduced equation are determined.

Note that omitting the last term (with the resonant frequency differences) in (3.5) and (3.9) violates necessary symmetry conditions for $\tilde{V}^{(2)}$ and $\tilde{W}^{(2)}$ which provide Hamiltoniaty of the reduced equations (2.23) and (2.25). But it can be proved that when the resonance conditions (2.27) and (2.28) are satisfied exactly (i.e. on the resonance surfaces themselves) these conditions are satisfied. It should also be emphasized that the above calculations do not impose any constraints on the smallness of the resonance frequency differences $\Delta_{0+1-2-3}$ and $\Delta_{0+1-2-3-4}$, and this essentially distinguishes our approach from those currently in use.

Consider now the possibility of the existence of 'integrals of motion' of the type

$$I = \int r(\boldsymbol{k})\, b^*(\boldsymbol{k})\, b(\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}. \tag{3.36}$$

For $r(\boldsymbol{k}) = \boldsymbol{k}$ the quantity I is the wave momentum, and for $r(\boldsymbol{k}) = 1$ it is the wave action. It follows from the five-wave reduced equation (2.25) that I evolves according to the equation

$$2\mathrm{i}\frac{\partial I}{\partial t} = \int (r_0+r_1-r_2-r_3)\, \tilde{V}^{(2)}_{0,1,2,3}\, b_0^* b_1^* b_2 b_3\, \delta_{0+1-2-3}\, \mathrm{d}k_{0123}$$
$$+\int (r_0+r_1-r_2-r_3-r_3)\, \tilde{W}^{(2)}_{0,1,2,3,4}(b_0^* b_1^* b_2 b_3 b_4 - b_0 b_1 b_2^* b_3^* b_4^*)\, \delta_{0+1-2-3-4}\, \mathrm{d}k_{01234}. \tag{3.37}$$

It is seen from this equation that the five-wave reduced equation (2.25) conserves the momentum only, but the four-wave reduced equation conserves both the momentum and the action. Thus, the five-wave interactions, described by the fourth-order terms in (2.25), violate the wave action conservation law intrinsic only to the four-wave interactions. Equation (3.37) with $r(\boldsymbol{k}) = 1$ permits us to estimate this effect quantitatively. Of course, both equations (2.23) and (2.25) conserve the reduced Hamiltonian (2.22) and (2.24) respectively.

We stress that for the derivation of equation (3.37) all symmetry properties of the kernels are required. If, for example, we use the kernel $\tilde{V}^{(2)}_{0,1,2,3} = Z^{(2)}_{0,1,2,3} + V^{(2)}_{0,1,2,3}$ discussed above instead of the kernel (3.5) or (3.25), then an equation like (3.37) cannot be derived, and the above conservation laws do not hold, as was noticed by Caponi *et al.* (1982).

4. Expansion of the Hamiltonian

This article would be incomplete without presenting the expansion coefficients of the Hamiltonian H, which enter practically all the above expressions. The expansion up to the fourth-order terms in a and a^* for the case of deep-water waves has been given by Zakharov (1968), but with a number of omissions. Most of the papers have derived an evolution equation for a like (2.13) from original hydrodynamical equations, without the Hamiltonian formulation and the related expansion of the Hamiltonian. As has been pointed out in the introduction, the coefficients of the evolution equation obtained this way usually do not satisfy the symmetry conditions expressing the Hamiltonian structure of the system. Here we present an expansion of H with accuracy up to the fifth-order terms for capillary–gravity waves on fluid of finite depth.

The general solution of the Laplace equation satisfying the bottom boundary condition can be presented by the following Fourier integral:

$$\phi(\boldsymbol{x}, z) = \frac{1}{2\pi}\int \phi(\boldsymbol{k})\frac{\cosh[|\boldsymbol{k}|(z+h)]}{\sinh(|\boldsymbol{k}|h)}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{k}, \quad \phi(\boldsymbol{k}) = \phi^*(-\boldsymbol{k}). \tag{4.1}$$

The calculations then proceed as follows: (1) expand the Hamiltonian H in powers of $\zeta(\boldsymbol{k})$ and $\phi(\boldsymbol{k})$ with accuracy up to the fifth-order terms; (2) express $\phi(\boldsymbol{k})$ through $\zeta(\boldsymbol{k})$ and $\psi(\boldsymbol{k})$ with accuracy up to the fourth-order terms; (3) present H through the canonically conjugate variables $\zeta(\boldsymbol{k})$ and $\psi(\boldsymbol{k})$ with accuracy up to the fifth-order terms; (4) present H through the normal variables $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$ with accuracy up to the fifth-order terms using the formulae (2.7). We turn now to realization of this plan and present the key points of the calculations.

(1) Turn first to the expression (2.2) for the kinetic energy K. Using (4.1) we find

$$\int_{-h}^{\zeta}\left[(\nabla\phi)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]\mathrm{d}z = -\frac{1}{(2\pi)^2}\int\int[(\boldsymbol{k}\cdot\boldsymbol{k}_1)I^+ - |\boldsymbol{k}|\,|\boldsymbol{k}_1|I^-]\phi(\boldsymbol{k})\,\phi(\boldsymbol{k}_1)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}_1)\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}_1,$$

where
$$I^s = \frac{n\sinh[m(h+\zeta)]+sm\sinh[n(h+\zeta)]}{mn[\cosh(mh)-\cosh(nh)]},\quad s=\pm,$$

with $m = |\boldsymbol{k}|+|\boldsymbol{k}_1|$, $n = |\boldsymbol{k}|-|\boldsymbol{k}_1|$.

Assuming $|\boldsymbol{k}|\zeta$ is small (weak nonlinearity), we replace the hyperbolic sines in I^s by their Taylor-series expansions up to orders $(m\zeta)^3$ and $(n\zeta)^3$, which is sufficient for presenting the kinetic energy with accuracy up to the fifth-order terms. Then, using (2.4) we present expansion of the kinetic energy in the form

$$\begin{aligned} K = {} & \frac{1}{2}\int|\boldsymbol{k}|\operatorname{cotanh}(|\boldsymbol{k}|h)\,\phi_0^*\,\phi_0\,\mathrm{d}\boldsymbol{k}_0 - \frac{1}{2(2\pi)}\int K_{0,1}^{(3)}\,\phi_0\,\phi_1\,\zeta_2\,\delta_{0+1+2}\,\mathrm{d}\boldsymbol{k}_{012} \\ & -\frac{1}{2(2\pi)^2}\int K_{0,1}^{(4)}\,\phi_0\,\phi_1\,\zeta_2\,\zeta_3\,\delta_{0+1+2+3}\,\mathrm{d}\boldsymbol{k}_{0123} \\ & -\frac{1}{2(2\pi)^3}\int K_{0,1}^{(5)}\,\phi_0\,\phi_1\,\zeta_2\,\zeta_3\,\zeta_4\,\delta_{0+1+2+3+4}\,\mathrm{d}\boldsymbol{k}_{01234}, \end{aligned} \tag{4.2}$$

with

$$K_{0,1}^{(3)} = \operatorname{cotanh}(|\boldsymbol{k}|h)\operatorname{cotanh}(|\boldsymbol{k}_1|h)[(\boldsymbol{k}\cdot\boldsymbol{k}_1)-q_0\,q_1],$$
$$K_{0,1}^{(4)} = \tfrac{1}{2}\operatorname{cotanh}(|\boldsymbol{k}|h)\operatorname{cotanh}(|\boldsymbol{k}_1|h)\{[(\boldsymbol{k}\cdot\boldsymbol{k}_1)-|\boldsymbol{k}_1|^2]\,q_0+[(\boldsymbol{k}\cdot\boldsymbol{k}_1)-|\boldsymbol{k}|^2]\,q_1\},$$
$$K_{0,1}^{(5)} = \tfrac{1}{6}\operatorname{cotanh}(|\boldsymbol{k}|h)\operatorname{cotanh}(|\boldsymbol{k}_1|h)[(|\boldsymbol{k}|^2+|\boldsymbol{k}_1|^2)(\boldsymbol{k}\cdot\boldsymbol{k}_1)-2|\boldsymbol{k}|^2|\boldsymbol{k}_1|^2-|\boldsymbol{k}-\boldsymbol{k}_1|^2q_0\,q_1].$$

In the above expressions $q(\boldsymbol{k})$ is given in (2.9) and we have used compact notation together with complete one where it is convenient.

The highest-order term in the expansion of the potential energy (2.3) is the fourth-order one (the next terms are already of the sixth order). This expansion is trivial and does not need special explanation:

$$\Pi = \frac{1}{2}\int\tau_0\,\zeta_0^*\,\zeta_0\,\mathrm{d}\boldsymbol{k}_0+\int\Pi_{0,1,2,3}^{(4)}\,\zeta_0\,\zeta_1\,\zeta_2\,\zeta_3\,\delta_{0+1+2+3}\,\mathrm{d}\boldsymbol{k}_{0123}, \tag{4.3}$$

with

$$\Pi_{0,1,2,3}^{(4)} = -\frac{\gamma}{24(2\pi)^2}[(\boldsymbol{k}\cdot\boldsymbol{k}_1)(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)+(\boldsymbol{k}\cdot\boldsymbol{k}_2)(\boldsymbol{k}_1\cdot\boldsymbol{k}_3)+(\boldsymbol{k}\cdot\boldsymbol{k}_3)(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)].$$

Here $\tau(\boldsymbol{k})$ is given in (2.9) and proper symmetrization has been made for $\Pi_{0,1,2,3}^{(4)}$. Thus, we have found the expansion of $H = K+\Pi$ in terms of the functions $\zeta(\boldsymbol{k})$ and $\phi(\boldsymbol{k})$ with accuracy up to the fifth-order terms.

(2) To express H in terms of $\zeta(\boldsymbol{k})$ and $\psi(\boldsymbol{k})$ one should first find the relation of $\phi(\boldsymbol{k})$ to $\zeta(\boldsymbol{k})$ and $\psi(\boldsymbol{k})$. It is clear from the structure of (4.2) for K that this relation should be found with accuracy up to the fourth-order terms. We have from (4.1)

$$\psi(\boldsymbol{x}) = \frac{1}{2\pi}\int\phi(\boldsymbol{k})\frac{\cosh[|\boldsymbol{k}|(\zeta+h)]}{\sinh(|\boldsymbol{k}|h)}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{k}, \tag{4.4}$$

with $\zeta = \zeta(\boldsymbol{x})$. This formula, together with (2.4) and (2.5), give the required relation.

To find this relation in explicit form the hyperbolic cosine in (4.4) should be replaced by a Taylor-series expansion up to order $(|\boldsymbol{k}|\zeta)^3$. Using this expansion and (2.4), (2.5) and (4.4) we obtain

$$\psi_0 = \operatorname{cotanh}(|\boldsymbol{k}|h)\,\phi_0 + \frac{1}{2\pi}\int |\boldsymbol{k}_1|\phi_1\,\zeta_2\,\delta_{0-1-2}\,\mathrm{d}k_{12}$$
$$+\frac{1}{(2\pi)^2}\int \tfrac{1}{2}|\boldsymbol{k}_1|^2 \operatorname{cotanh}(|\boldsymbol{k}_1|h)\,\phi_1\,\zeta_2\,\zeta_3\,\delta_{0-1-2-3}\,\mathrm{d}k_{123}$$
$$+\frac{1}{(2\pi)^3}\int \tfrac{1}{6}|\boldsymbol{k}_1|^3\phi_1\,\zeta_2\,\zeta_3\,\zeta_4\,\delta_{0-1-2-3-4}\,\mathrm{d}k_{1234}. \tag{4.5}$$

This equation should be solved by iterations relative to $\phi(\boldsymbol{k})$ with accuracy up to the fourth-order terms in $\zeta(\boldsymbol{k})$ and $\psi(\boldsymbol{k})$. The solution, after proper symmetrization, is

$$\phi_0 = \tanh(|\boldsymbol{k}|h)\Bigg[\psi_0 - \frac{1}{2\pi}\int q_1\,\psi_1\,\zeta_2\,\delta_{0-1-2}\,\mathrm{d}k_{12} - \frac{1}{(2\pi)^2}\int \Phi^{(3)}_{0,1,2,3}\,\psi_1\,\zeta_2\,\zeta_3\,\delta_{0-1-2-3}\,\mathrm{d}k_{123}$$
$$-\frac{1}{(2\pi)^3}\int \Phi^{(4)}_{0,1,2,3,4}\,\psi_1\,\zeta_2\,\zeta_3\,\zeta_4\,\delta_{0-1-2-3-4}\,\mathrm{d}k_{1234}\Bigg], \tag{4.6}$$

where

$$\Phi^{(3)}_{0,1,2,3} = \tfrac{1}{2}(|\boldsymbol{k}_1|^2 - q_1\,q_{0-2} - q_1\,q_{0-3}),$$
$$\Phi^{(4)}_{0,1,2,3,4} = \tfrac{1}{6}[q_1(|\boldsymbol{k}_1|^2 - |\boldsymbol{k}_1+\boldsymbol{k}_2|^2 - |\boldsymbol{k}_1+\boldsymbol{k}_3|^2 - |\boldsymbol{k}_1+\boldsymbol{k}_4|^2)$$
$$-|\boldsymbol{k}_1|^2(q_{0-2}+q_{0-3}+q_{0-4}) + q_1\,q_{0-2}(q_{1+4}+q_{1+3})$$
$$+q_1\,q_{0-3}(q_{1+4}+q_{1+2}) + q_1\,q_{0-4}(q_{1+3}+q_{1+2})].$$

(3) We can now present H in terms of the canonically conjugate variables $\zeta(\boldsymbol{k})$ and $\psi(\boldsymbol{k})$ with accuracy up to the fifth-order terms. The potential energy is expressed through $\zeta(\boldsymbol{k})$ only. So one should transform the kinetic energy by substituting (4.6) into (4.2) and retaining therein the terms up to the fifth order. After cumbrous algebra with numerous cancellations one obtains relatively simple result:

$$K = \frac{1}{2}\int q_0\,\psi_0^*\,\psi_0\,\mathrm{d}k_0 + \int E^{(3)}_{0,1,2}\,\psi_0\,\psi_1\,\zeta_2\,\delta_{0+1+2}\,\mathrm{d}k_{012}$$
$$+\int E^{(4)}_{0,1,2,3}\,\psi_0\,\psi_1\,\zeta_2\,\zeta_3\,\delta_{0+1+2+3}\,\mathrm{d}k_{0123} + \int E^{(5)}_{0,1,2,3,4}\,\psi_0\,\psi_1\,\zeta_2\,\zeta_3\,\zeta_4\,\delta_{0+1+2+3+4}\,\mathrm{d}k_{01234}, \tag{4.7}$$

where

$$E^{(3)}_{0,1,2} = -\frac{1}{2(2\pi)}[(\boldsymbol{k}\cdot\boldsymbol{k}_1) + q_0\,q_1],$$

$$E^{(4)}_{0,1,2,3} = -\frac{1}{8(2\pi)^2}[2|\boldsymbol{k}|^2 q_1 + 2|\boldsymbol{k}_1|^2 q_0 - q_0\,q_1(q_{0+2}+q_{1+2}+q_{0+3}+q_{1+3})],$$

$$E^{(5)}_{0,1,2,3,4} = -\frac{1}{12(2\pi)^3}\{2|\boldsymbol{k}|^2|\boldsymbol{k}_1|^2 + \tfrac{1}{2}q_0\,q_1(|\boldsymbol{k}|^2 + |\boldsymbol{k}_1|^2 - |\boldsymbol{k}+\boldsymbol{k}_2|^2$$
$$-|\boldsymbol{k}+\boldsymbol{k}_3|^2 - |\boldsymbol{k}+\boldsymbol{k}_4|^2 - |\boldsymbol{k}_1+\boldsymbol{k}_2|^2 - |\boldsymbol{k}_1+\boldsymbol{k}_3|^2 - |\boldsymbol{k}_1+\boldsymbol{k}_4|^2)$$
$$-|\boldsymbol{k}_1|^2 q_0(q_{0+2}+q_{0+3}+q_{0+4}) - |\boldsymbol{k}|^2 q_1(q_{1+2}+q_{1+3}+q_{1+4})$$
$$+q_0\,q_1[q_{0+2}(q_{1+3}+q_{1+4}) + q_{0+3}(q_{1+2}+q_{1+4}) + q_{0+4}(q_{1+2}+q_{1+3})]\}.$$

In the case of deep-water waves one should just replace $q(\boldsymbol{k})$ by $|\boldsymbol{k}|$ in the above formulae.

At this stage of the calculations, the pair of canonically conjugate Hamilton equations (2.6) for $\zeta(\boldsymbol{k})$ and $\psi(\boldsymbol{k})$ can be written in the explicit form:

$$\frac{\partial \zeta_0}{\partial t} - q_0 \psi_0 = 2\int E^{(3)}_{-0,1,2}\psi_1\zeta_2\delta_{0-1-2}\,\mathrm{d}k_{12} + 2\int E^{(4)}_{-0,1,2,3}\psi_1\zeta_2\zeta_3\delta_{0-1-2-3}\,\mathrm{d}k_{123}$$
$$+2\int E^{(5)}_{-0,1,2,3,4}\psi_1\zeta_2\zeta_3\zeta_4\delta_{0-1-2-3-4}\,\mathrm{d}k_{1234}, \quad (4.8)$$

$$\frac{\partial \psi_0}{\partial t} + \tau_0\zeta_0 = -\int E^{(3)}_{1,2,-0}\psi_1\psi_2\delta_{0-1-2}\,\mathrm{d}k_{12} - 2\int E^{(4)}_{1,2,3,-0}\psi_1\psi_2\zeta_3\delta_{0-1-2-3}\,\mathrm{d}k_{123}$$
$$-4\int \Pi^{(4)}_{1,2,3,-0}\zeta_1\zeta_2\zeta_3\delta_{0-1-2-3}\,\mathrm{d}k_{123} - 3\int E^{(5)}_{1,2,3,4,-0}\psi_1\psi_2\zeta_3\zeta_4\delta_{0-1-2-3-4}\,\mathrm{d}k_{1234}, \quad (4.9)$$

where the surface tension is also included.

In some respects, this system of equations seems more convenient for numerical study of combined four- and five-wave interactions than the reduced equation (2.25). Firstly, these equations have simpler kernels, and, secondly, they give $\zeta(\boldsymbol{k})$ directly rather than the auxiliary variable $b(\boldsymbol{k})$, which is related to $\zeta(\boldsymbol{k})$ through $a(\boldsymbol{k})$ by a complicated canonical transformation. Note that the kernels on the right-hand sides of these equations are related, in any given order, to each other through the same coefficients $E^{(n)}$ by quite definite way, expressing the Hamiltonian structure of these equations. To obtain such a structure from original hydrodynamical equations is practically impossible.

(4) Finally, it remains to express the Hamiltonian H in terms of the normal variable $a(\boldsymbol{k})$ and obtain the coefficients $U^{(n)}$, $V^{(n)}$, $W^{(n)}$ in expansion (2.11). To do this, one should just substitute (2.7) into H expressed above through $\zeta(\boldsymbol{k})$ and $\psi(\boldsymbol{k})$. After proper symmetrication, we obtain the following expressions:

$$U^{(1)}_{0,1,2} = -U_{-0,1,2} - U_{-0,2,1} + U_{1,2,-0}, \quad U^{(3)}_{0,1,2} = U_{0,1,2} + U_{0,2,1} + U_{1,2,0},$$

$$V^{(1)}_{0,1,2,3} = \tfrac{1}{3}(-V_{-0,1,2,3} - V_{-0,2,1,3} - V_{-0,3,1,2} + V_{1,2,-0,3} + V_{1,3,-0,2} + V_{2,3,-0,1}) - 4\Gamma_{0,1,2,3},$$

$$V^{(2)}_{0,1,2,3} = V_{-0,-1,2,3} + V_{2,3,-0,-1} - V_{-0,2,-1,3} - V_{-1,2,-0,3} - V_{-0,3,-1,2} - V_{-1,3,-0,2} + 12\Gamma_{0,1,2,3},$$

$$V^{(4)}_{0,1,2,3} = \tfrac{1}{3}(V_{0,1,2,3} + V_{0,2,1,3} + V_{0,3,1,2} + V_{1,2,0,3} + V_{1,3,0,2} + V_{2,3,0,1}) + 4\Gamma_{0,1,2,3},$$

$$W^{(1)}_{0,1,2,3,4} = \tfrac{1}{2}(W_{1,2,3,4,-0} + W_{1,3,2,4,-0} + W_{1,4,2,3,-0} + W_{2,3,1,4,-0} + W_{2,4,1,3,-0} + W_{3,4,1,2,-0} - W_{-0,1,2,3,4} - W_{-0,2,1,3,4} - W_{-0,3,1,2,4} - W_{-0,4,1,2,3}),$$

$$W^{(2)}_{0,1,2,3,4} = 2(W_{-0,-1,2,3,4} - W_{-0,2,-1,3,4} - W_{-1,2,-0,3,4} - W_{-0,3,-1,2,4} - W_{-1,3,-0,2,4} - W_{-0,4,-1,2,3} - W_{-1,4,-0,2,3} + W_{2,3,-0,-1,4} + W_{2,4,0,-1,3} + W_{3,4,-0,-1,2}),$$

$$W^{(5)}_{0,1,2,3,4} = \tfrac{1}{2}(W_{0,1,2,3,4} + W_{0,2,1,3,4} + W_{0,3,1,2,4} + W_{0,4,1,2,3} + W_{1,2,0,3,4} + W_{1,3,0,2,4} + W_{1,4,0,2,3} + W_{2,3,0,1,4} + W_{2,4,0,1,3} + W_{3,4,0,1,2}),$$

where $U_{0,1,2} = -\mathcal{N}_0\mathcal{N}_1\mathcal{M}_2 E^{(3)}_{0,1,2}, \quad V_{0,1,2,3} = -2\mathcal{N}_0\mathcal{N}_1\mathcal{M}_2\mathcal{M}_3 E^{(4)}_{0,1,2,3},$

$$\Gamma_{0,1,2,3} = \mathcal{M}_0\mathcal{M}_1\mathcal{M}_2\mathcal{M}_3\Pi^{(4)}_{0,1,2,3}, \quad W_{0,1,2,3,4} = -\mathcal{N}_0\mathcal{N}_1\mathcal{M}_2\mathcal{M}_3\mathcal{M}_4 E^{(5)}_{0,1,2,3,4}.$$

The functions $U^{(n)}$, $V^{(n)}$, $W^{(n)}$ satisfy all necessary symmetry conditions.

5. Discussion

This paper was stimulated by publications which discuss, in some way or another, the non-Hamiltonian structure of reduced equations for surface gravity waves, though the problem of constructing the Hamiltonian reduced equations is of general importance for weakly nonlinear waves in conservative dispersive media allowing a Hamiltonian description (see e.g. the review by Zakharov 1974). The canonical transformation technique solves the problem in a natural way, resulting in Hamiltonian reduced equations.

In view of the fact that the non-Hamiltonian reduced equations remain in use and many interesting results have already been obtained based on them, it seems necessary to estimate their accuracy as compared to the Hamiltonian ones. The most direct way of such an estimation consists in a comparison of consequences from non-Hamiltonian and Hamiltonian reduced equations. Such a comparison for the case of the four-wave reduced equation applied to the deep-water gravity waves was recently given by Krasitskii & Kalmykov (1993). Two typical examples were considered: the modulational instability of a uniform Stokes wave train and the long-time evolution of a discrete wave system. Overall, one can draw the conclusion that distinctions in solutions of the reduced equations in the non-Hamiltonian and the Hamiltonian forms are revealed only for sufficiently large nonlinearity of the wave system. It should be noted that the non-Hamiltonian form of the reduced equations do not give any simplifications in either analytical or numerical analysis. Thus it should not be used.

Consider now some statistical aspects associated with the reduced equations and the canonical transformation. Define the 'observable' wavenumber spectrum $F(\boldsymbol{k})$ of a horizontally uniform random wave field by

$$F(\boldsymbol{k}) = \frac{1}{(2\pi)^2}\frac{\omega(\boldsymbol{k})}{g}N(\boldsymbol{k}), \quad \langle a(\boldsymbol{k})\, a^*(\boldsymbol{k}')\rangle = N(\boldsymbol{k})\,\delta(\boldsymbol{k}=\boldsymbol{k}'),$$

where the angle brackets imply an ensemble average. The observable spectrum is normalized by the condition $\langle \zeta^2\rangle = \int F(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k}$ and has the property $F(\boldsymbol{k}) \neq F(-\boldsymbol{k})$, in view of which it is called non-symmetric. Note that the function $N(\boldsymbol{k})$ coincides, correct to the factor $(2\pi)^2 g$, with the spectral wave action $F(\boldsymbol{k})/\omega(\boldsymbol{k})$. By analogy with $F(\boldsymbol{k})$ and $N(\boldsymbol{k})$ we define the functions $f(\boldsymbol{k})$ and $n(\boldsymbol{k})$ by

$$f(\boldsymbol{k}) = \frac{1}{(2\pi)^2}\frac{\omega(\boldsymbol{k})}{g}n(\boldsymbol{k}), \quad \langle b(\boldsymbol{k})\, b^*(\boldsymbol{k}')\rangle = n(\boldsymbol{k})\,\delta(\boldsymbol{k}-\boldsymbol{k}').$$

We call $f(\boldsymbol{k})$ and $n(\boldsymbol{k})$ weak-interaction spectra to distinguish them from observable ones. Using the usual quasi-Gaussian closure one can derive from the five-wave reduced equation (2.25) the following kinetic (transfer) equation for $n(\boldsymbol{k})$:

$$\begin{aligned}\frac{\partial n_0}{\partial t} = {} & 4\pi\int [\tilde{V}^{(2)}_{0,1,2,3}]^2 n_0 n_1 n_2 n_3\left[\frac{1}{n_0}+\frac{1}{n_1}-\frac{1}{n_2}-\frac{1}{n_3}\right] \\ & \times \delta(\omega_0+\omega_1-\omega_2-\omega_3)\,\delta_{0+1-2-3}\,\mathrm{d}\boldsymbol{k}_{123} \\ & + 12\pi\int [\tilde{W}^{(2)}_{0,1,2,3,4}]^2 n_0 n_1 n_2 n_3 n_4\left[\frac{1}{n_0}+\frac{1}{n_1}-\frac{1}{n_2}-\frac{1}{n_3}-\frac{1}{n_4}\right] \\ & \times \delta(\omega_0+\omega_1-\omega_2-\omega_3-\omega_4)\,\delta_{0+1-2-3-4}\,\mathrm{d}\boldsymbol{k}_{1234} \\ & + 18\pi\int [\tilde{W}^{(2)}_{4,3,2,1,0}]^2 n_0 n_1 n_2 n_3 n_4\left[\frac{1}{n_0}+\frac{1}{n_1}+\frac{1}{n_2}-\frac{1}{n_3}-\frac{1}{n_4}\right] \\ & \times \delta(\omega_0+\omega_1+\omega_2-\omega_3-\omega_4)\,\delta_{0+1+2-3-4}\,\mathrm{d}\boldsymbol{k}_{1234}+\ldots. \end{aligned} \tag{5.1}$$

The first integral on the right-hand side of (5.1) corresponds to the four-wave kinetic equation derived by Hasselmann (1962), and the two other integrals describe spectral energy transfer due to the five-wave interactions; the dots at the end of the equation represent fourth-order terms in n, not written down here, originating from cubic in b terms in the reduced equation (2.25) and, thus, depending on $\tilde{V}^{(2)}$.

Consider the possibility of the existence of conservation laws of the form $J = \int r(\boldsymbol{k})\, n(\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}$ for equation (5.1). For $r = \boldsymbol{k}$ the quantity J is proportional to the mean momentum of the random wave field, for $r = \omega(\boldsymbol{k})$ it is proportional to the mean potential energy and for $r = 1$ it is proportional to the mean wave action (all the quantities refer to unit horizontal area). It is easy to find from (5.1) that the quantity J evolves according to the equation

$$\frac{\partial J}{\partial t} = \pi \int [\tilde{V}^{(2)}_{0,1,2,3}]^2 [r_0 + r_1 - r_2 - r_3]\, n_0 n_1 n_2 n_3 \left[\frac{1}{n_0} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}\right]$$
$$\times\, \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3)\, \delta_{0+1-2-3}\, \mathrm{d}\boldsymbol{k}_{0123}$$
$$+ 6\pi \int [\tilde{W}^{(2)}_{0,1,2,3,4}]^2 [r_0 + r_1 - r_2 - r_3 - r_4]\, n_0 n_1 n_2 n_3 n_4 \left[\frac{1}{n_0} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4}\right]$$
$$\times\, \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3 - \omega_4)\, \delta_{0+1-2-3-4}\, \mathrm{d}\boldsymbol{k}_{01234}. \tag{5.2}$$

It is seen from this equation that $\partial J/\partial t = 0$ for $r = \boldsymbol{k}$ and for $r = \omega(\boldsymbol{k})$, i.e. the five-wave kinetic equation conserves the momentum and the energy (of course, these conservation laws should be valid for any number of interacting waves). At the same time, the first integral in (5.2) vanishes for $r = 1$, and the second one does not vanish, i.e. the four-wave kinetic integral conserves the wave action, and the five-wave one leads to violation of this conservation law.

At the same time, many important inferences from the four-wave kinetic equation are connected just with the wave action conservation law, in particular the inference about the existence of a Kolmogorov power-law spectrum caused by constant action flux from the short-wave to long-wave range of the spectrum (Zakharov 1984). Thus taking account of five-wave interactions, violating the wave action conservation law, does not just improve precision, related to allowing for higher-order terms in perturbation theory, but is of principal significance. Evolution of the action is described by (5.2) with $r = 1$.

Usually the difference between $N(\boldsymbol{k})$ and $n(\boldsymbol{k})$ is either ignored and these spectra are simply identified, or else is not mentioned. In practical applications we need the observable spectrum $F(\boldsymbol{k})$, so we have to consider its relationship to $f(\boldsymbol{k})$.

We can find this relationship using the canonical transformation (2.17) and a statistical hypothesis similar to that employed in the derivation of the kinetic equation. Using (2.17), we have to calculate the correlation function $\langle a(\boldsymbol{k})\, a^*(\boldsymbol{k}')\rangle$, and apply the Gaussian hypothesis to the correlation functions of higher orders in b, which appear on the right-hand side of the equation. This calculation procedure yields

$$N_0 = n_0 + 4n_0 \int B^{(2)}_{0,1,0,1}\, n_1\, \mathrm{d}\boldsymbol{k}_1$$
$$+ 2\int \{[A^{(1)}_{0,1,0-1}]^2 n_1 n_{0-1} + 2[A^{(1)}_{0+1,0,1}]^2 n_1 n_{0+1} + [A^{(3)}_{0,1,-0-1}]^2 n_1 n_{-0-1}\}\, \mathrm{d}\boldsymbol{k}_1 + \dots, \tag{5.3}$$

where the dots at the end of the equation represent cubic and fourth-order terms in n, not written down here, originating from corresponding terms in the canonical transformation (2.17).

In practical applications (5.3) calls for numerical calculations. However, some simple conclusions can be drawn even using its general structure. First, the power-law spectra $n(\boldsymbol{k})$ (for example, the Kolmogorov spectra which are just weak-turbulence ones) when transformed to the observable spectra $N(\boldsymbol{k})$, are no longer of the power-law type, at least because of the complex dependence of $A^{(n)}$ on $\boldsymbol{k}$. Secondly, in the case of spectra $n(\boldsymbol{k})$ which are narrow in the $\boldsymbol{k}$-space and concentrated, say, in the vicinity of the wave vector $\boldsymbol{k}_0$, the spectra $N(\boldsymbol{k})$ exhibit additional 'secondary' peaks at $\boldsymbol{k}\pm 2\boldsymbol{k}_0$, $\boldsymbol{k}\pm 3\boldsymbol{k}_0$, $\boldsymbol{k}\pm 4\boldsymbol{k}_0$ (the latter are due to the cubic and the fourth-order terms in n which are not included in (5.3)). This is easily seen in the limiting case of a monochromatic wave, when $n(\boldsymbol{k}) = c\delta(\boldsymbol{k}-\boldsymbol{k}_0)$, $c = \text{const}$, and (5.3) yields

$$N(\boldsymbol{k}) = c\delta(\boldsymbol{k}-\boldsymbol{k}_0)+4c^2B^{(2)}(\boldsymbol{k}_0,\boldsymbol{k}_0,\boldsymbol{k}_0,\boldsymbol{k}_0)\,\delta(\boldsymbol{k}-\boldsymbol{k}_0)$$
$$+2c^2\{[A^{(1)}(2\boldsymbol{k}_0,\boldsymbol{k}_0,\boldsymbol{k}_0)]^2\delta(\boldsymbol{k}-2\boldsymbol{k}_0)+[A^{(3)}(-2\boldsymbol{k}_0,\boldsymbol{k}_0,\boldsymbol{k}_0)]^2\delta(\boldsymbol{k}+2\boldsymbol{k}_0)\}+\dots.$$

The nature of these peaks is essentially the same as in the pioneering work by Tick (1959), who seemingly was the first to carry out perturbation analysis of a random sea surface up to the second-order terms. Such secondary peaks are frequently observed in the measured spectra of wind waves in the ocean (in this case $\boldsymbol{k}_0$ is the wave vector of the main maximum in the spectrum). The angular energy distribution of $N(\boldsymbol{k})$ and $n(\boldsymbol{k})$ is also different. Roughly speaking, the distinctions between $N(\boldsymbol{k})$ and $n(\boldsymbol{k})$ are of the order of the square of spectral component steepness and can be seen (as numerical estimates for model wind wave spectra have shown) only in a short-wave range of the spectrum, far from the spectral maximum (so these distinctions can be neglected when describing energy-containing spectral components of wind waves in the ocean).

To show the connection of our fourth-order depth-dependent equations (4.8) and (4.9) with other known approximate equations (and to check, to some extent, the coefficients in these equations) we consider the limit of shallow-water weakly dispersive waves. We approximate the function q_0 on the left-hand side of (4.8), describing dispersion of linear waves, by the expression $q(\boldsymbol{k}) \approx |\boldsymbol{k}|^2h-\frac{1}{3}|\boldsymbol{k}|^4h^3$; in the coefficients $E^{(3)}$ and $E^{(4)}$ we use the approximation $q(\boldsymbol{k}) \approx |\boldsymbol{k}|^2h$; and in the coefficient $E^{(5)}$ we let $q(\boldsymbol{k}) = 0$. In this case we have

$$E^{(3)}_{0,1,2} = -\frac{1}{2(2\pi)}[(\boldsymbol{k}\cdot\boldsymbol{k}_1)+|\boldsymbol{k}|^2|\boldsymbol{k}_1|^2h^2],$$

$$E^{(4)}_{0,1,2,3} = -\frac{1}{2(2\pi)^2}|\boldsymbol{k}|^2|\boldsymbol{k}_1|^2h, \quad E^{(5)}_{0,1,2,3,4} = -\frac{1}{6(2\pi)^3}|\boldsymbol{k}|^2|\boldsymbol{k}_1|^2.$$

In this approximation, (4.8) and (4.9) in coordinate representation correspond to the 'Boussinesq-like evolution equation'

$$\frac{\partial\zeta}{\partial t} = -\nabla\cdot(\tilde{h}\nabla\psi)-\tfrac{1}{3}\nabla^2(\tilde{h}^3\nabla^2\psi) = \frac{\delta H}{\delta\psi},$$

$$\frac{\partial\psi}{\partial t} = -g\zeta+\gamma\{\nabla^2\zeta-\tfrac{1}{2}\nabla\cdot[(\nabla\zeta)^2\nabla\zeta]\}-\tfrac{1}{2}(\nabla\psi)^2+\tfrac{1}{2}(\tilde{h}\nabla^2\psi)^2 = -\frac{\delta H}{\delta\zeta},$$

where now $\zeta = \zeta(\boldsymbol{x},t)$, $\psi = \psi(\boldsymbol{x},t)$, $\tilde{h} = h+\zeta$ is the depth beneath the displaced surface, and

$$H = \frac{1}{2}\int\{g\zeta^2+\gamma[(\nabla\zeta)^2-\tfrac{1}{4}(\nabla\zeta)^4]+\tilde{h}(\nabla\psi)^2-\tfrac{1}{3}\tilde{h}^3(\nabla^2\psi)^2\}\,\mathrm{d}\boldsymbol{x}.$$

This expression for H is obtained using (4.3), (4.7), and the above approximations for $E^{(n)}$.

These Boussinesq-like equations have been derived by other methods by a number of authors, e.g. Miles & Salmon (1985) (see also references therein). Note that in the considered limit of shallow-water weakly dispersive waves the Hamiltonian H in coordinate representation is local functional of ζ and ψ, in contrast with the general case of arbitrary depth in which H is a non-local functional of these variables, as Miles (1977) has pointed out.

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